Matched interface and boundary (MIB) method for multi-domain elliptic interface problems

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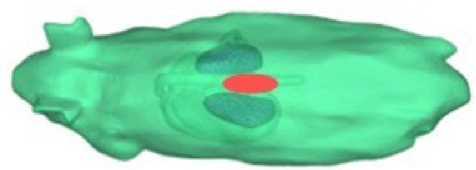
Introduction

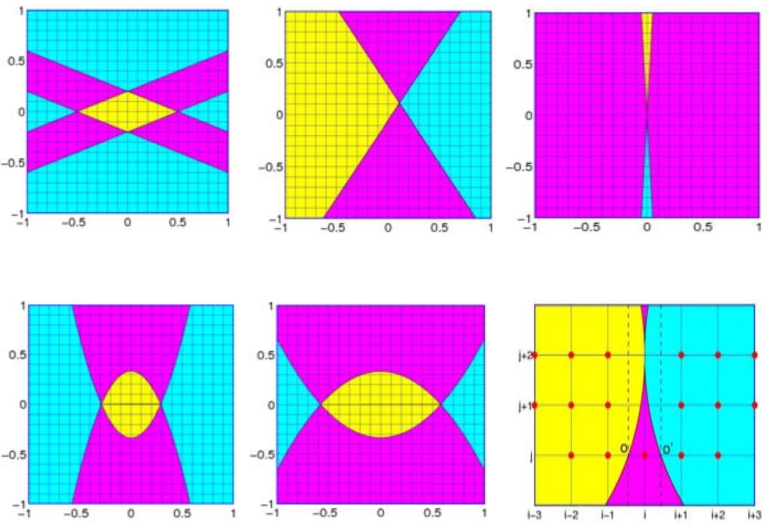
Multidomain interface problems are omnipresent in science, engineering and daily life. These problems become exceptionally challenging when more than two heterogeneous materials join at a point of the space and form a singularity. For instant, two different internal organs and the internal liquid can form a three-domain singularity (or triple junction). This work develops the matched interface boundary (MIB) method to solve elliptic equations with three-domain discontinuous coefficients and singular sources, as well as singular interface geometry in two dimensions.



Theory

The MIB method splits the 2D problem into 1D ones, enforces the lowest order interface conditions across interfaces and utilizes the standard central finite difference for the discreteization of the governing equation. Fictitious values are obtained on irregular points near the interfaces to replace the real ones.

In the three domain situation, different geometric shapes are studied. Real challenges come from the geometric singularities. New schemes are created to solve different singularities, such as triple junctions, multiple junctions, sharp tips, etc.



Algorithm

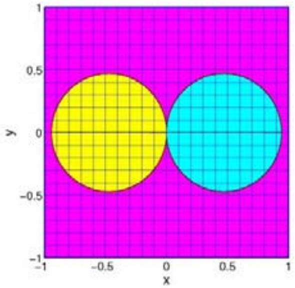
$$[u]_o = (w^{ao}_{0,i-2}, w^{ao}_{0,i-1}, w^{ao}_{0,i}) \cdot (u_{i-2,j}, u_{i-1,j}, f^{ao}_{i,j})^T - (w^{co}_{0,i-1}, w^{co}_{0,i}, w^{co}_{0,i+1}) \cdot (f^{co}_{i-1,j}, u_{i,j}, f^{co'}_{i+1,j})^T$$

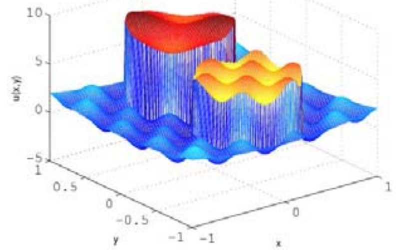
$$\begin{aligned}[\beta u_n]_o - \beta^a \tan\theta [u_\tau]_o = \; & C^{ao}_x \begin{bmatrix} w^{ao}_{1,i-2} \\ w^{ao}_{1,i-1} \\ w^{ao}_{1,i} \end{bmatrix}^T \cdot \begin{bmatrix} u_{i-2,j} \\ u_{i-1,j} \\ f^{ao}_{i,j} \end{bmatrix} - C^{co}_x \begin{bmatrix} w^{co}_{1,i-1} \\ w^{co}_{1,i} \\ w^{co}_{1,i+1} \end{bmatrix}^T \cdot \begin{bmatrix} f^{co}_{i-1,j} \\ u_{i,j} \\ f^{co'}_{i+1,j} \end{bmatrix} - C^{co}_y \begin{bmatrix} w_{1,j} \\ w_{1,j+1} \\ w_{1,j+2} \end{bmatrix}^T \\ & \cdot \begin{bmatrix} w^{ao}_{0,i-2} & w^{ao}_{0,i-1} & w^{ao}_{0,i} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w1^{ao}_{0,i-3} & w1^{ao}_{0,i-2} & w1^{ao}_{0,i-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w2^{ao}_{0,i-3} & w2^{ao}_{0,i-2} & w2^{ao}_{0,i-1} \end{bmatrix} \\ & \cdot \left[u_{i-2,j}, u_{i-1,j}, f^{ao}_{i,j}, u_{i-3,j+1}, u_{i-2,j+1}, u_{i-1,j+1}, u_{i-3,j+2}, u_{i-2,j+2}, u_{i-1,j+2}\right]^T \end{aligned}$$

Similar two equations can be get by enforcing the jump conditions at o' point and then the four fictitious values can be represented in terms of real function values and jump conditions. The FD schemes are:

$$\begin{aligned} (\beta u_x)_x &= \frac{1}{\Delta x^2}(\beta^a_{i-\frac{3}{2},j}, -\beta^a_{i-\frac{3}{2},j} - \beta^a_{i-\frac{1}{2},j}, \beta^a_{i-\frac{1}{2},j}) \cdot (u_{i-2,j}, u_{i-1,j}, f^{ao}_{i,j})^T \quad \text{at}(i-1,j) \\ (\beta u_x)_x &= \frac{1}{\Delta x^2}(\beta^c_{i-\frac{1}{2},j}, -\beta^c_{i-\frac{1}{2},j} - \beta^c_{i+\frac{1}{2},j}, \beta^c_{i+\frac{1}{2},j}) \cdot (f^{co}_{i-1,j}, u_{i,j}, f^{co'}_{i+1,j})^T \quad \text{at}(i,j) \\ (\beta u_x)_x &= \frac{1}{\Delta x^2}(\beta^b_{i+\frac{1}{2},j}, -\beta^b_{i+\frac{1}{2},j} - \beta^b_{i+\frac{3}{2},j}, \beta^b_{i+\frac{3}{2},j}) \cdot (f^{bo'}_{i,j}, u_{i+1,j}, u_{i+2,j})^T \quad \text{at}(i+1,j) \end{aligned}$$

Numerical studies





$$\nabla \cdot (\beta \nabla u(x,y)) = q(x,y)$$

$$\beta^a = 1, \beta^b = 2, \beta^c = 3$$

$$\Omega^a : u(x,y) = 6 + \sin(4\pi x)\sin(4\pi y)$$

$$\Omega^b : u(x,y) = 8 + \sin(2\pi x)\sin(2\pi y)$$

$$\Omega^c : u(x,y) = x^2 + y^2 + sin(4\pi x)sin(4\pi y)$$

$n_x \times n_y$	L_∞	L_2	Order(L_∞)	Order(L_2)
20*20	5.1717e-1	1.0974e-1		
40*40	9.4894e-2	1.9957e-2	2.45	2.46
80*80	2.0840e-2	4.7404e-3	2.19	2.07
160*160	4.1347e-3	1.0739e-3	2.33	2.14

References

K. Xia, M. Zhan and G.W. Wei, The MIB method for multidomain elliptic interface problems, J. Comput. Phys., To be submitted.

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